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Fermions Coupled to Chern-Simons Gauge Field or Imaginary Chemical Potential and the Bloch Theorem

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Abstract: I point out that the $U(N)$ Chern-Simons $3d$ theory coupled to fermions at finite temperature and at a specific mean field approximation and the $3d$ Gross-Neveu model at finite temperature and imaginary chemical potential can give us the same results for the thermodynamic values of the free-energy and the saddle point equation for the thermal mass. I use specific results from the thermodynamics of fermionic models that coupled to Chern-Simons gauge field and imaginary chemical potential. In the latter case I introduce a representation for the canonical partition function for imaginary chemical potential and I see that the CS level κ plays the role of the $U(1)$ charge. I further argue that the periodic structure of the imaginary chemical potential brings also Bloch's theorem into the game. Namely, the vacuum structure of the fermionic system with imaginary baryon density is a Bloch wave. I further emphasise that Bloch waves correspond to fermionic (antisymmetric) or bosonic (symmetric) quasi-particles depending on the point in the band one sits in. This situation is similar with particles in a periodic potential of a crystal that behave like Bloch-wavefunctions. The overlap between them is a lattice momentum that can be restricted to the first Brillouin zone of the band structure.

Keywords: Fermions, Chern-Simons, Bloch-wave

1. Introduction

In recent years, the study of topological field theories has gained significant attention due to their intriguing mathematical structures and potential applications in high-energy physics and condensed matter physics like the particle-vortex duality [1, 2], the fermion-boson duality [3] and bosonization phenomena [4, 5]. Among these theories, Chern-Simons gauge theory coupled to fermions has emerged as a fascinating framework for investigating exotic phenomena and novel phases of matter (for thermal Chern-Simons coupled to matter theories see for example [6, 7] and references therein and the more recent work for large charge at large N [8]). In this paper, I explore the interplay between Chern-Simons gauge theory and fermions at finite temperature, with a particular focus on the relationship with a fermionic theory at imaginary chemical potential.

I begin by providing a brief overview of Chern-Simons gauge theory and its relevance in various physical systems. I discuss the fundamental aspects of fermions coupled to Chern-Simons gauge fields, emphasizing their role in generating topological terms and inducing fractional statistics. Next, I introduce the concept of a finite-temperature formalism for the coupled system and present the necessary tools to describe the thermodynamics and transport properties of the theory and at the end I give new view into the partition function of a fermionic theory at imaginary chemical potential as a Bloch wave.

Overall, this paper provides a comprehensive investigation into the interplay between Chern-Simons gauge theory, fermions and imaginary chemical potential at finite temperature. My results shed light on the rich physics underlying this coupled system and offer insights into potential applications in various areas of physics, including topological insulators, fractional quantum Hall systems and high-energy physics.

2. General Notation

2.1. Maxwell

In order to compare a $4d$ Maxwell theory with a $3d$ Chern-Simons theory, coupled to matter, I begin with electromagnatism. Let's start with Maxwell's equations

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\,,\,\vec{\nabla}\vec{E}=\frac{\rho}{\epsilon_0} \tag{1}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{j}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\,,\,\vec{\nabla}\vec{B}=0 \tag{2}$$

and from now on I set $\epsilon_0\mu_0=1/c^2$, $c=1$. If one wants to present a relativistic theory then sets

$$\eta_{\mu\nu}=diag(-1,1,1,1)\,,\;\; i,j,k,..=1,2,3\,,\;\mu,\nu,\rho,..=0,1,2,3 \tag{3}$$

Then the field strength tensor is

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{4}$$

If one wants to write the Maxwell theory coupled to a source of matter (fermion or boson) then sets

$$I=I_M+I_{int}=\int d^4x\mathcal{L}_M+\int d^4xA_\mu J^\mu=-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}+\int d^4xA_\mu J^\mu \tag{5}$$

which leads to the equation of motion

$$\partial_\mu F^{\mu\nu}=-J^\mu \tag{6}$$

If one defines

$$\vec{E}\mapsto E^i=F^{0i}\,,\;\;\vec{B}\mapsto B^i=\frac{1}{2}\epsilon^{ijk}F_{jk}\Rightarrow F_{ij}=\epsilon_{ijk}B^k \tag{7}$$

Notice that $\epsilon_{123}=\epsilon^{123}=1$. With the above notations I find

$$I_M=\frac{1}{2}\int d^4x(\vec{E}^2-\vec{B}^2) \tag{8}$$

and comparing with Maxwell's equations

$$J^\mu=\left(\frac{\rho}{\epsilon_0},\mu_0\vec{j}\right) \tag{9}$$

2.2. Abelian Chern-Simons

The Abelian CS action coupled to sources is

$$I=I_{CS}+I_{int}=\int d^3x\mathcal{L}_{CS}+I_{int}=\frac{\kappa}{2}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+\int d^3xA_\mu J^\mu \tag{10}$$

Notice that now we are in three dimensions hence

$$\eta_{\mu\nu}=diag(-1,1,1)\,,\;\; i,j,k,..=1,2\,,\;\;\mu,\nu,\rho,..=0,1,2 \tag{11}$$

where now $\epsilon_{012}=-\epsilon^{012}=1$. Obviously the Chern-Simons action does not seem to depend on the metric so we can write it in a way that

$$\int_{\mathcal{N}}A\wedge dA=\int_{\mathcal{N}}(A_\mu\partial_\nu A_\rho)dx^\mu\wedge dx^\nu\wedge dx^\rho \tag{12}$$

where $\mathcal{N}$ is a $3d$ Lorentzian manifold which does not have a boundary. Recall now that if one calculates the invariant measure on $\mathcal{N}$ is

$$dV_3=\epsilon^0\wedge e^1\wedge e^2=\sqrt{-g}dx^0\wedge dx^1\wedge dx^2=\sqrt{-g}d^3x \tag{13}$$

Notice now that

$$dx^\mu \wedge dx^\nu \wedge dx^\rho = \epsilon^{\mu\nu\rho} dx^0 \wedge dx^1 \wedge dx^2 = E^{\mu\nu\rho} dV_3 \tag{14}$$

It bears significance to highlight that $\epsilon_{\mu\nu\rho}$ takes values ± 1 and since

$$\epsilon^{\mu\nu\rho} = \frac{1}{(\sqrt{-g})^2} \epsilon_{\mu\nu\rho} \tag{15}$$

it is *not* a tensor. The proper tensor density is $E_{\mu\nu\rho}$ defined as

$$E^{\mu\nu\rho} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho} \tag{16}$$

Since the Chern-Simons action is topological, we see from the above that the metric does not appear in it.
Let's go back to the Chern-Simons action where the equations of motion are

$$\frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = -J^\mu \ , \ \ \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = -J^\mu \tag{17}$$

With the preview definitions these read

$$B = \frac{\rho}{\kappa} \ , \ \ \epsilon^{ij} E_j = \frac{1}{\kappa} J^i \ , \ \ J^\mu = (\rho, J^i) \tag{18}$$

The new condition now is that the magnetic field B is a pseudoscalar.

2.3. Non-Abelian Chern-Simons

The gauge potential now has a transformation in the adjoint representation of the complex $SU(N)$ Lie algebra

$$A_\mu = A^a T^a \ , \ \ [T^a, T^b] = f^{abc} T^c \ , \ \ Tr(T^a T^b) = \frac{1}{2} \delta^{ab} \ , \ \ a, b, c, ... = 1, 2, ... N^2 - 1 \tag{19}$$

The prime example is $SU(2)$ where $T^a = \frac{1}{2} \sigma^a$ and the structure constants f^{abc} are complex. The non-abelian action is

$$\mathcal{I}_{CS} = \frac{\kappa}{4\pi} \int_{\mathcal{N}} Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = \frac{\kappa}{4\pi} \int_{\mathcal{N}} d^3x \epsilon^{\mu\nu\rho} Tr \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right)$$

Since under the gauge transformation (g is a group element of $SU(N)$)

$$A \mapsto g^{-1} A g + g^{-1} dg \tag{20}$$

the CS action is not invariant but changes as

$$\mathcal{I}_{CS} \mapsto \mathcal{I}_{CS} - 2\pi\kappa \int_{\mathcal{N}} w(g) \tag{21}$$

where $w(g)$ is the winding number

$$w(g) = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho} Tr \left(g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g \right) \tag{22}$$

then, invariance of the partition function

$$\mathcal{Z} = \int (\mathcal{D} A_\mu) e^{i\mathcal{I}_{CS}} \tag{23}$$

requires that κ is an integer $\kappa = N$, $N \in \mathbb{Z}$.

3. The Fractional Quantum Hall Effect in Connection with Chern-Simons Gauge Field Coupled to Matter

The Lagrangian of a Chern-Simons theory when fermionic or bosonic matter is coupled with the gauge field is

$$L_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \tag{24}$$

The Euler-Lagrange equations are (17) where J^μ is the current of bosonic or fermionic matter $\left(\rho, J^i\right)$ and $i = 1, 2$ in $3d$. There are 3 components of the Euler-Lagrange equations with the first component

$$\rho = \kappa B \tag{25}$$

This equation indicates that the local charge density is directly proportional to the magnetic field, implying that a Chern-Simons field intertwines magnetic flux with electric charge, giving rise to $anyons$. So imagine a scenario with a group of N particles linked with magnetic flux Φ. The crucial idea here is that each particle perceives all the other $N - 1$

particles as individual point vortices carrying a flux of Φ. This model proves suitable for describing the fractional quantum Hall effect, a physical phenomenon characterized by quantized plateaus in the Hall conductance of 2D electrons, exhibiting fractional values of e^2/h. The clarity of this assertion emerges through calculating the overall phase resulting from the interaction between the charge e of an $anyon$ and the total flux of the other $anyons$ revolving around it in an adiabatic way. Assuming N particles encircling this $anyon$, the phase of the $anyon$ changes by $2\pi N$. Introducing the Berry phase of this exchange as Γ, we obtain the following

$$\Gamma = 2\pi N \tag{26}$$

However, when dealing with these N particles we assume that each particle carries a flux of Φ and since the value of the flux quantum is $\Phi_0 = \frac{h}{e}$ then

$$\Gamma = 2\pi N \frac{\Phi}{\Phi_0} \tag{27}$$

It is interesting to remember past works where the calculations were conducted while considering that the eigenvalues of the Chern-Simons gauge field existed within the thermal circle and exhibited periodic characteristics, akin to electrons encircling an $anyon$ [7]. This $anyon$-eigenvalue emerged as a consequence of the earlier mentioned flux attachment. Remarkably, the eigenvalues of the Chern-Simons field mimicked electron behavior, and the model's effective action described these eigenvalues as particles moving in a magnetic field. Consequently, their quantization occurred in discrete units of $\frac{e}{\kappa}$, κ represents the Chern-Simons level and fill ν hypothetic Landau levels. The total Γ phase is now

$$\Gamma = 2\pi N \frac{\frac{e}{\kappa}}{\Phi_0} \tag{28}$$

where again $\Phi_0 = \frac{h}{e}$. Having in mind that $\hbar = \frac{h}{2\pi}$ then the overall phase turns to

$$\Gamma = N \frac{e^2}{\hbar\kappa} \tag{29}$$

For $\hbar = 1$ the Γ phase is now [9]

$$\Gamma = N \frac{e^2}{\kappa} \tag{30}$$

So when an electron moves adiabatically around another electron its wavefunction acquires a phase like

$$\psi' = e^{iN\frac{e^2}{\kappa}} \psi \tag{31}$$

or by using the 't Hooft coupling $\lambda = \frac{N}{\kappa}$ we have

$$\psi' = e^{iN\frac{\lambda e^2}{N}} \psi \tag{32}$$

and for simplicity if we set $e^2 = 1$ we have finally

$$\psi' = e^{i\lambda} \psi \tag{33}$$

In this scenario, we can visualize electrons encircling a thin solenoidal magnetic flux, leading us to anticipate the occurrence of phenomena reminiscent of the Bohm-Aharonov effect [10, 11]. A similar picture is to have fermions at imaginary chemical potential as we will examine later.

4. Imaginary Chemical Potential vs Chern-Simons Gauge Field

Returning to equation (25) one may think the possibility to use an electric field instead of a magnetic one to achieve an electric flux attachment this time. We have in mind that in $2+1$ dimensions physics coming from the coupling of a Chern-Simons gauge theory with fermionic matter, the two components of the electric field lie on the same plane in contrast with the one component of the magnetic field which is perpendicular to the plane. So we may suppose that we form a fermionic model at a real chemical potential that lies on the same plane as the Chern-Simons electric components. Then one may rotates it until it comes on the imaginary axis and the model turns to a fermionic model with imaginary chemical potential where we can study statistical transmutation but also some amazing similarities with the above models. Since physics with an electromagnetic field like the Chern-Simons field is much more complicated than physics with an imaginary electric potential, our main ambition is to examine the condition:

"Gross Neveu model at critical values of the imaginary chemical potential→ Fractional quantum Hall effect at critical values of the ν Landau levels".

Of course a model with a chemical potential is closed to a model with an electric field in a crystal and polarization phenomena from the Berry phase coming from the periodic structure of the crystal. So if one imagines the movement of an electron in a crystal (along for example the first Brillouin zone) a neighbor electron somehow travels around it because a Brillouin zone in one dimension can be mapped onto a circle, in view of the fact that wavevectors $k = 0$ and $k = \frac{2\pi}{a}$ label the same states.

An equivalent picture is the above: When fermions are coupled to a Chern-Simons gauge field in a monopole background, their system exhibits intriguing properties such as the emergence of anyonic statistics, which refer to statistics that are neither fermionic nor bosonic but can be fractional. The study of theories like these finds applications in condensed matter physics, like the FQHE and topological insulators.

At finite temperature, it has been postulated that in three Euclidean dimensions, when considering Dirac fermions coupled to an abelian Chern-Simons field at level κ [1], the presence of a monopole charge is closely related to the case when an imaginary chemical potential is introduced into the system [12, 13]. A similar work at higher odd dimensions appeared in [14, 15]. Let's see the connection:

[1] My notations follow [12].

$$Z_{fer}(\beta,\kappa)=\int[\mathcal{D}A_\nu][\mathcal{D}\bar{\psi}][\mathcal{D}\psi]\exp\left[-S_{fer}(\bar{\psi},\psi,A_\nu)\right], \tag{34}$$

$$S_{fer}(\bar{\psi},\psi,A_\nu)=-\int_0^\beta d\tau\int d^2\bar{x}\left[\bar{\psi}(\partial\!\!\!/-iA\!\!\!/)\psi+i\frac{\kappa}{4\pi}\epsilon_{\nu\lambda\rho}A_\nu\partial_\lambda A_\rho+...\right]. \tag{35}$$

There are also some fermionic self interactions that their presence presented with dots. When one expands the CS field around time independent monopole configuration $\bar{A}_\nu$ [16]

$$A_\nu=\bar{A}_\nu+b_\nu\,,\ \ \bar{A}_\nu=(0,\bar{A}_1(\bar{x}),\bar{A}_2(\bar{x}))\,,\ \ b_\nu=(b_0(\tau),b_1(\tau,\bar{x}),b_2(\tau,\bar{x}))\,, \tag{36}$$

which is normalized as[2]

$$\frac{1}{2\pi}\int d^2x\bar{F}_{12}=1\,,\ \ \bar{F}_{\nu\lambda}=\partial_\nu\bar{A}_\lambda-\partial_\lambda\bar{A}_\nu \tag{37}$$

and b_ν is a backround gauge field.

Hence, (35) describes the possibility of monopole configurations in the fermionic theory (or the case involving the attachment of κ units of monopole charge to the fermions) as

$$S_{fer}(\bar{\psi},\psi,A_\nu)=-\int_0^\beta d\tau\int d^2\bar{x}\left[\bar{\psi}(\partial\!\!\!/-i\gamma_i\bar{A}_i-i\gamma_\nu b_\nu)\psi+i\frac{\kappa}{4\pi}\epsilon_{\nu\lambda\rho}b_\nu\partial_\lambda b_\rho+..\right]-i\kappa\int_0^\beta d\tau b_0\,. \tag{38}$$

We can perform the path integral over the CS fluctuations like viewing the theory with fixed total monopole charge. To do this, I use a mean field approximation in this sector where the spatial CS fluctuations balance out the magnetic background gauge filed $\langle b_i\rangle=-\bar{A}_i$ [17].[3] It is comperable to how we may think of a reduction in which the background gauge field's integral along the thermal circle is fixed to be a constant. One obtains

$$\begin{aligned}Z_{fer}(\beta,\kappa)&=\int[\mathcal{D}b_0][\mathcal{D}\bar{\psi}][\mathcal{D}\psi]\exp\left[\int_0^\beta d\tau\int d^2\bar{x}\left[\bar{\psi}(\partial\!\!\!/-i\gamma_0 b_0)\psi+..\right]+i\kappa\int_0^\beta dx^0 b_0\right]\\&=\int(\mathcal{D}\theta)e^{i\kappa\theta}Z_{gc,fer}(\beta,-i\theta/\beta)\,,\end{aligned} \tag{39}$$

where $\theta=\int_0^\beta d\tau b_0(\tau)$, $Z_{gc,fer}(\beta,-i\theta/\beta)$ is the grand canonical partition function for the fermionic theory and I have used standard formulae from [12]. Observing the situation, it becomes evident that the Chern-Simons level κ assumes the role of the eigenvalue Q of the $U(1)$ charge operator. Here is an important conclusion that the finite temperature partition function of Dirac fermions coupled to abelian CS gauge field at level κ in a monopole background, can be equivalently represented as the canonical partition function of the fermions with a fixed fermion number κ. This discussion shows that the partition function of fermions coupled to abelian CS in a monopole background is intimately related to the respective canonical partition function at fixed total $U(1)$ charge.

Let's compare the two theories by using previous results. Here we have a theory of N Dirac fermions ψ^i $(i=1,....,N)$ coupled to $U(N)$ Chern-Simons gauge field A_μ. The theory has a definition described by the Lagrangian

$$L_{CS(f)}=\overline{\psi}^iD_\mu\psi^i+\sigma\overline{\psi}^i\psi^i+\frac{i\kappa}{4\pi}\varepsilon^{\mu\nu\rho}Tr(A_\mu\partial_\nu A_\rho+\frac{2}{3}A_\mu A_\nu A_\rho) \tag{40}$$

where $D_\mu\psi=\partial_\mu\psi-iA^aT^a\psi$.

When the theory is critical (The Gross-Neveu model with Chern-Simons interactions at the saddle point), the primary scalar operator denoted by σ, assumes a crucial role in the theory, necessitating its inclusion in the path integral computation. A previous work was made by using the light-cone gauge to eliminate the non-abelian part of the Lagrangian [7]. Since in thermodynamics the fermions have anti-periodic boundary conditions on the thermal circle, the momentum on the circle is quantized as $p_3=\frac{2\pi}{\beta}(n+\frac{1}{2}), n\in\mathbb{Z}$.

The free-energy density of the system is

$$\frac{F_{CS}}{V_2}=\frac{N}{2\pi\beta^3}\left\{\frac{\mu_F^3}{3}\left(1\mp\frac{1}{\lambda}\right)+\frac{\widetilde{\sigma}\mu_F^2}{2\lambda}-\frac{\widetilde{\sigma}^3}{6\lambda}+\frac{1}{\pi i\lambda}\int_{\mu_F}^\infty dyy\left[Li_2\left(-e^{-y+\pi i\lambda}\right)-c.c.\right]\right\} \tag{41}$$

where $\widetilde{\sigma}=\sigma\beta$ and $\lambda=\frac{N}{\kappa}$ is the 't Hooft coupling. When the theory is critical we extremize the $\frac{F_{CS}}{V_2}$ with respect to $\widetilde{\sigma}$ and we find

[2] It is interesting to put the theory on $S^1\times S^2$.

[3] It would be interesting to further explain whether an appropriate large-N is also necessary for the validity of such an approximation.

$$\widetilde{\sigma} = \pm\mu_F \tag{42}$$

With the appropriate choice of signs (the (-) sign in the parenthesis and the (+) sign at the above equation) the critical free-energy density we obtain is

$$\frac{F_{CS}}{V_2} = \frac{N}{2\pi\beta^3}\left\{\frac{\mu_F^3}{3} + \frac{1}{\pi i\lambda}\int_{\mu_F}^{\infty} dyy\left[Li_2\left(-e^{-y+\pi i\lambda}\right) - c.c.\right]\right\} \tag{43}$$

and the saddle point equation is

$$\lambda\mu_F = -\frac{1}{\pi i}\left[Li_2\left(-e^{-\mu_F-\pi i\lambda}\right) - c.c.\right] \tag{44}$$

The main difference here with the $2+1$ Gross-Neveu model in the canonical formalism from previous calculations is to include the integral over all the possible A_0 values ($A_0 = \frac{2\pi nc}{e}$ where $c\epsilon\,[-1/2, 1/2]$)[13]. The corresponding result is

$$F_{GN} = \frac{NV_2}{2\pi}\left\{\frac{m^3}{3} + \int_0^{\infty} pdp\left(\frac{1}{i\pi\left(\frac{n}{e}\right)}\right)\left[Li_2\left(-e^{-L\sqrt{p^2+m^2}+i2\pi\left(\frac{n}{e}\right)}\right) - Li_2\left(-e^{-L\sqrt{p^2+m^2}-i2\pi\left(\frac{n}{e}\right)}\right)\right]\right\} \tag{45}$$

Obviously the expressions of the free-energies of the two models are the same if we consider that

$$\mu_F = m\beta \tag{46}$$

and

$$\lambda = \frac{n}{e} \tag{47}$$

and also

$$m = -\frac{1}{\pi i\left(\frac{n}{e}\right)}\left[Li_2\left(-e^{-m\beta-\pi i\left(\frac{n}{e}\right)}\right) - c.c.\right] \tag{48}$$

In principle, the free-energy F_{GN} together with the gap equation are sufficient for investigating the thermodynamic properties of the Gross-Neveu model up to leading order in N.

However the main conclusion from the above analysis is that as the Chern-Simons gauge field "ties" charge with flux by the Chern-Simons κ level and somehow we have fixed number of particles, the constraint that we insert at the Gross-Neveu model tells us that we have also fixed number of particles. One may assume this from the beginning of our analysis by examine carefully the Lagrangians of the models and find the equivalence of the parts.

$$L_1 = i\kappa b_0 \tag{49}$$

which is a (flux)$\times$(gauge potential as magnetic field) and

$$L_2 = iB\theta \tag{50}$$

which is also a (electric flux arises from the charge of particles)$\times$(potential) [13]. So we may say that with the constraint we put in the Lagrangian we "create" something like a topological gauge field.

5. Partition Function and Bloch Waves

The grand canonical partition function of a Dirac fermion[4] in $3d$ and at imaginary chemical potential can be written as

$$Z^f_{gc}(\beta, i\theta/\beta) = \int(\mathcal{D}\psi)(\mathcal{D}\bar{\psi})e^{-\int_0^\beta d\tau\int d^2\vec{x}\left[\bar{\psi}(\gamma^\mu\partial_\mu - i\gamma^0\theta/\beta)\psi + V(\bar{\psi}\psi)\right]} \tag{51}$$

where momentarily we are not interested in the potential term $V(\bar{\psi}\psi)$. Placing the fermions at finite temperature requires imposing antiperiodic boundary conditions along the compactified Euclidean time τ as

$$\bar{\psi}(\beta, \bar{x}) = -\bar{\psi}(0, \bar{x})\,,\;\; \psi(\beta, \bar{x}) = -\psi(0, \bar{x}) \tag{52}$$

When there is an imaginary chemical potential in (51) is actually equivalent to the coupling of the fermions to a background gauge potential of the form $A_\mu = (\theta/\beta, 0, 0)$. One therefore might think that this is removable by a simple gauge transformation, and setting

$$\bar{\psi}(\tau, \bar{x}) \mapsto \bar{\psi}'(\tau, \bar{x}) = e^{-i\int_0^\tau d\tilde{\tau}\alpha_0(\tilde{\tau})}\bar{\psi}(\tau, \bar{x})\,, \tag{53}$$

$$\psi(\tau, \bar{x}) \mapsto \psi'(\tau, \bar{x}) = e^{i\int_0^\tau d\tilde{\tau}\alpha_0(\tilde{\tau})}\psi(\tau, \bar{x})\,. \tag{54}$$

could do the job. However, such a transformation would twist the antiperiodic boundary conditions (52) since now we would obtain

$$\bar{\psi}(\beta, \bar{x}) = -e^{i\theta}\bar{\psi}(0, \bar{x})\,,\;\; \psi(\beta, \bar{x}) = -e^{-i\theta}\psi(0, \bar{x}) \tag{55}$$

[4] For simplicity in this section I consider theories with a single fermion or complex scalar.

where these twists are intimately related to the confining/deconfining of colour singlets. Obviously there is a connection with (33) and this is a main result of this work. Also, if one consider ψ as a Bloch wave then θ could change the "lattice" periodic part of the wavefunction.

Then I discuss the theory for imaginary chemical potential. The important point is that this situation corresponds to having the GN model coupled to a $U(1)$ potential fluctuating around its $\mathbb{Z}$-vacua. This is the proposal of my previous work [13]. The fact that I use an imaginary chemical potential brings the Lee-Yang theorem into the game. We need to elaborate on that, in particular since I find *negative* free-energy densities. This shows that we probe unstable critical points. But this is not the main case in this work.

This case resembles to investigations of quantum mechanical systems in a periodic potential akin to a periodic crystal. If we conceive of θ as a periodic coordinate, then it corresponds to computing the overlap between two Bloch wavefunctions that vary by lattice momentum B [18]. Typically, such systems yield a band structure that can be analysed by confining the lattice momentum to the first Brillouin zone.

Consider a system in three dimensions and at finite temperature $T = 1/\beta$ with a global $U(1)$ charge operator $\hat{Q}$. Its canonical partition function can be formally calculated as the thermal average over states with fixed $\hat{Q}$ as

$$Z_c(\beta, B) = Tr\left[\delta(\hat{Q} - B)e^{-\beta\hat{H}}\right] \tag{56}$$

If the eigenvalues B of $\hat{Q}$ are integers, namely if the system contains elementary excitations, an explicit representation of (56) can be written as

$$Z_c(\beta, B) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta B}\, \mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{\mathrm{H}} - \mathrm{i}\theta\hat{\mathrm{Q}}}\right] = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{\mathrm{i}\theta\mathrm{B}}\, \mathrm{Z_{gc}}(\beta, \mathrm{i}\mu = \mathrm{i}\theta/\beta), \tag{57}$$

where $Z_{gc}(T, i\mu)$ is the grand canonical partition function with imaginary chemical potential $i\mu$.

In the simple systems we are interested in one expects that

$$Z_{gc}(\beta, i(\mu + 2\pi k/\beta)) = Z_{gc}(\beta, i\mu)\,, \quad k \in \mathbb{Z} \tag{58}$$

One then notices with Bloch's theorem as follows. In quantum system, taken here to be $1d$ for clarity, in a periodic potential with period a the energy eigenstates are the Bloch waves

$$\psi_k(x) = e^{ikx}u(x)\,, \quad u(x+a) = u(x) \tag{59}$$

where k is the lattice momentum vector. The transition amplitude between two Bloch waves with different lattice momenta is

$$Z(k_2 - k_1) \equiv \langle\psi_{k_1}|\psi_{k_2}\rangle = \int_0^a dx e^{i(k_2-k_1)x}|u(x)|^2 \tag{60}$$

Notice the formal equivalence of (60) with (57), which implies that B may be thought of as a transfer momentum when a Bloch wave scatters from a lattice point [19]. A suitable expression of a $1d$ Bloch-wave is of the form:

$$\psi_k = e^{ikx}u(x) \tag{61}$$

where u function has the periodicity of the lattice a like $Z_{gc}(\beta, i\mu = i\theta/\beta)$ has the periodicity of the chemical potential.

6. Conclusion

I have pointed out that the canonical partition function of the Gross-Neveu model at finite temperature and imaginary chemical potential is intricately connected to the thermal partition function of abelian Chern-Simons fields coupled to matter in a monopole background, particularly when employing a suitable mean-field approximation. In this context, the system can be viewed as being in a regime where the $U(1)$ charge density essentially aligns with the monopole charge. Motivated from my past works about the relationship between the partition function of the fermionic model and the Bloch theorem I shed more light to the connection between the periodic structure of the imaginary chemical potential on the thermal circle and a periodic band structure where a Bloch wave travels [19]. It would also be interesting in the future to examine the specific values of the imaginary chemical potential where there are phase transitions of the model and their relationship with the FQHE.

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